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HYPERPLANE SECTIONS OF ABELIAN SURFACES

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ABSTRACT. By a theorem of Wahl, the canonically embedded curves which are hyperplane section of K3 surfaces are distinguished by the non-surjectivity of their Wahl map. In this paper we address the problem of distinguishing hyperplane sections of abelian surfaces. The somewhat surprising result is that the Wahl map of such curves is (tendentially) surjective, but their *second* Wahl map has corank at least 2 (in fact a more precise result is proved).

1. Introduction

Which canonically embedded curves $C \subset \mathbb{P}^{g-1}$ are hyperplane sections of K3 surfaces? An answer to this question is provided by Wahl's theorem ([17],[18] see also the proof by Beauville-Mérindol in [2]), which asserts that, if C is a curve sitting on a $K3$ surface, then its Wahl map, or gaussian map,

$$\gamma^1_C : \wedge^2 H^0(K_C) \to H^0(K_C^3)$$

is not surjective. The significance of this criterion is better appreciated if one compares it with two other results:

- Ciliberto-Harris-Miranda's theorem ([3], see also Voisin's proof in [16]), stating that the Wahl map of the generic curve of genus g is surjective as soon as this is numerically possible, i.e. for $g \geq 10$, with the exception of $g = 11$. For $g < 10$ and $g = 11$ it is known that the generic curve lies on a K3 surface ([12]).

- Lazarsfeld theorem ([10], see also [13] for a different proof), asserting (loosely speaking) that general hyperplane sections of general polarized $K3$ surfaces satisfy the Brill-Noether-Petri condition. Hence curves lying on K3 surfaces cannot be distinguished by special Brill-Noether-theoretic properties. In fact it has been independently conjectured by various authors (Mukai, Voisin, see [16]4.13(b), and Wahl, see [19]§0) that for Brill-Noether-Petri-general curves the non-surjectivity of the Wahl map should completely characterize curves contained in K3 surfaces.

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The aim of this note is to address the same questions for the other class of canonically embedded hyperplane sections of smooth surfaces: hyperplane sections of abelian surfaces. This case is somewhat more subtle since here the canonical embedding $C \hookrightarrow \mathbb{P}^{g-3}$ is not complete, as it is obtained by the complete canonical embedding in $\mathbb{P}^{g-1} = \mathbb{P} H^1(\mathcal{O}_C)$ after projection from the line $\mathbb{P}H^1(\mathcal{O}_X) \subset \mathbb{P}H^1(\mathcal{O}_C)$ (where X is the abelian surface). This is the first-order infinitesimal counterpart of the obvious intrinsic restriction satisfied by a curve sitting on an abelian surface, namely that its Jacobian is non-simple, as it contains the abelian surface $\mathrm{Pic}^0 X$. However, given a canonically embedded curve $C \subset \mathbb{P}^{g-3}$, obtained by projection from a line L in $\mathbb{P}^{g-1}$, it is not obvious how to recognize that its jacobian contains a two-dimensional abelian surface Y, so that L is the projectivized tangent space to Y. This is one of the features of our main result.

In order to set the stage, let us recall that the Wahl map γ^1 belongs to a hierarchy of maps, called higher gaussian maps. The second gaussian map, or second Wahl map, is a linear map

$$\gamma^2_C : I_2(C) \to H^0(K_C^4)$$

where $I_2(C)$ is the kernel of the natural map $S^2H^0(K_C) \to H^0(K_C^2)$ i.e. (if C is non-hyperelliptic) the vector space of quadrics of $\mathbb{P}^{g-1} = \mathbb{P}\ H^1(\mathcal{O}_C)$ containing C. The map γ^2_C has an independent interest, which was the original motivation of the first two authors for studying this sort of questions. In fact there is a relation, analyzed in [5], between the second gaussian map and the curvature of the moduli space M_g of curves of genus g, endowed with the Siegel metric induced by the period map $j : M_g \to A_g$. To be precise, in [5] the holomorphic sectional curvature of M_g along the a Schiffer variation ξ_P, for P a point on the curve C, was computed in terms of the holomorphic sectional curvature of A_g and the map γ^2_C. This was accomplished using the formula for the second fundamental form associated to the period map given in [7].

Going back to our problem, we introduce the following notation. Given a subspace $W \subset H^0(K_C)$, we will denote

$$S^2W \cdot H^0(K_C^2)$$

the image of $S^2W \otimes H^0(K_C^2)$ in $H^0(K_C^4)$ via the natural multiplication map. If W is 2-dimensional and base-point-free, the base-point-free pencil trick implies that $S^2W \cdot H^0(K_C^2)$ has codimension 2 in $H^0(K_C^4)$. If C is embedded in abelian surface, then $H^0(\Omega^1_X)$ is naturally a (base-point-free) 2-dimensional subspace of $H^0(K_C)$. Our main result is

Theorem A. *Let C be a curve contained in abelian surface X. Then the image of the second gaussian map γ^2_C is contained in $S^2H^0(\Omega^1_X) \cdot H^0(K_C^2)$ (notation as above). Therefore the corank of γ^2_C is at least 2.*

Moreover, if the second gaussian map of the surface X (see §2) *is surjective, then the image of the map* γ_C^2 *concides with* $S^2H^0(\Omega_X^1)\cdot H^0(K_C^2)$.

The above Theorem can be stated, perhaps more suggestively, as follows. Given a subspace $V \subset H^1(\mathcal{O}_C)$, let $\overline{V} \subset H^0(K_C)$ be its conjugate.

Corollary. *Let* $C \subset \mathbb{P}^{g-3}$ *be a canonically embedded curve of genus* g*, obtained from the complete canonical embedding* $C \subset \mathbb{P}H^1(\mathcal{O}_C) = \mathbb{P}^{g-1}$ *by projection from a line* $\mathbb{P}V \subset \mathbb{P}H^1(\mathcal{O}_C)$*,* $\dim V = 2$*. If* C *is a hyperplane section of an abelian surface* $X \subset \mathbb{P}^{g-2}$ *then*

$$Im(\gamma_C^2) \subseteq S^2\overline{V}\cdot H^0(K_C^2)$$

A few comments are in order. In the first place, the *first* gaussian map of a curve C sitting on an abelian surface X is "tendentially surjective". This is the content of another result proved in this note (we refer to Theorem 3.3 for the precise statement):

Theorem B. *Assume that the first gaussian map of the line bundle* $\mathcal{O}_X(C)$ *on the surface* X *is surjective and that the multiplication map*

$$\gamma^0_{X,C} : H^0(X, \mathcal{O}_X(C)) \otimes H^0(C, K_C) \to H^0(K_C^2)$$

is surjective (for example, both conditions hold if $\mathcal{O}_X(C)$ is at least a 5-th power of a ample line bundle on X, see [14]). *Then the first gaussian map of* C *is surjective.*

Secondly, one expects a Ciliberto-Harris-Miranda's theorem for second gaussian maps, namely that, for the generic curve of genus $g \geq 18$, the second gaussian map γ^2 is surjective. In [4] the first two authors exhibited infinitely many genera where this happens, by producing examples of curves lying on the product of two curves with surjective second Gaussian map whose second Gaussian map is surjective. Other examples were given in [1]. Both classes of examples generalize constructions given by Wahl for the first Gaussian map in [18], [17]. Moreover the first two authors ([6]) have proved the surjectivity of γ^2 for the generic curve of high genus (for $g > 152$).

Finally, concerning the Brill-Noether theory of curves on abelian surfaces, M. Paris ([15]) has obtained the following almost complete extension of Lazarsfeld's result: let X be an abelian surface such that its Néron-Severi group $NS(X)$ is cyclic, spanned by $c_1(L)$, with L an ample line bundle on X. Then all line bundles of degree $d \neq g(C) - 1$ on a general curve $C \in |L|$ satisfy the Petri condition. Hence, at least in degree different from $g - 1$, there are no Brill-Noether-theoretic ways to distinguish curves sitting on abelian surfaces. It is reasonable to conjecture that – under the hypothesis of sufficient Brill-Noether generality – the conclusion of the Corollary should characterize hyperplane section of abelian surfaces.

The proofs of both Theorems A and B are based on cohomological computations concerning the extension classes of the cotangent sequence

$$0 \to K_C^{-1} \to \Omega^1_{X|C} \to K_C \to 0$$

and of its "symmetric square"

$$0 \to \Omega^1_{X|C} \otimes K_C^{-1} \to S^2\Omega^1_{X|C} \to K_C^2 \to 0$$

Our approach has its roots in Beauville-Mérindol's paper ([2]).

Finally, we remark that, beyond Wahl's theorem, the geometric significance of Wahl's map is now reasonably well understood, thanks to the work of many authors (notably Voisin, [16]). On the other hand, Theorem A above, as well as the works [4],[5], seem to indicate that also the second gaussian map encodes some interesting geometry, which is at present much less understood.

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2. Preliminaries on gaussian maps

2.1. **Classical gaussian maps (of any order).** Let Y be a smooth complex projective variety and let $\Delta_Y \subset Y \times Y$ be the diagonal. Let L and M be line bundles on Y. For a non-negative integer k, the *k-th Gaussian map* associated to these data is the restriction map to the diagonal

(1)

$$\gamma^k_{L,M} : H^0(Y \times Y, I^k_{\Delta_Y} \otimes L \boxtimes M) \to H^0(Y, I^k_{\Delta_Y|\Delta_Y} \otimes L \otimes M) \cong H^0(Y, S^k\Omega^1_Y \otimes L \otimes M).$$

Usually *first* gaussian maps are simply referred to as *gaussian maps*. The exact sequence

$$0 \to I^{k+1}_{\Delta_Y} \to I^k_{\Delta_Y} \to S^k\Omega^1_Y \to 0, \tag{2}$$

(where $S^k\Omega^1_Y$ is identified to its image via the diagonal map), twisted by $L \boxtimes M$, shows that the domain of the k-th gaussian map is the kernel of the previous one:

$$\gamma^k_{L,M} : ker\gamma^{k-1}_{L,M} \to H^0(S^k\Omega^1_Y \otimes L \otimes M).$$

In our applications, we will exclusively deal with gaussian maps of order one and two, assuming also that the two line bundles L and M coincide. For the reader's convenience, we spell out these maps. The map γ^0_L is the multiplication map of global sections

$$H^0(X, L) \otimes H^0(X, L) \to H^0(X, L^2) \tag{3}$$

which obviously vanishes identically on $\wedge^2 H^0(L)$. Consequently, $H^0(Y \times Y, I_{\Delta_Y} \otimes L \boxtimes L)$ decomposes as $\wedge^2 H^0(L) \oplus I_2(L)$, where $I_2(L)$ is the kernel of $S^2H^0(X, L) \to H^0(X, L^2)$. Since γ^1_L vanishes on symmetric tensors, one writes

$$\gamma^1_L : \wedge^2 H^0(L) \to H^0(\Omega^1_Y \otimes L^2). \tag{4}$$

Again, $H^0(Y \times Y, I^2_{\Delta_Y} \otimes L \boxtimes L)$ decomposes as the sum of $I_2(L)$ and the kernel of (4). Since γ^2_L vanishes identically on skew-symmetric tensors, one usually writes

$$\gamma^2_L : I_2(L) \to H^0(S^2\Omega^1_Y \otimes L^2) \tag{5}$$

(In general, gaussian maps of even (resp.odd) order vanish identically on skew-symmetric (resp. symmetric) tensors). The primary object of this paper will be the *first and second order Wahl maps*, which are the first and second gaussian maps of the canonical line bundle K_C on a curve C:

$$\gamma^1_C : \wedge^2 H^0(K_C) \to H^0(K^3_C)$$
$$\gamma^2_C : I_2(K_C) \to H^0(K^4_C)$$

2.2. **Curve-surface gaussian maps.** In general, given a variety Y, endowed with a divisor Z on Y, it is useful to consider a variant of gaussian maps, which lies somewhat in between the gaussian maps on Y and the ones on Z. When Y is a surface and C is a curve on it, for easy reference we will sometimes call them *curve-surface gaussian maps*. These maps already appear in [16]. They are simply defined as follows: let L be a line bundle on Y, and let M_Z be a line bundle on Z, seen as a sheaf on Y. The k-th order gaussian map associated to these data is

$$\gamma^k_{L,M_Z} : H^0(Y \times Y, I^k_{\Delta_Y} \otimes L \boxtimes M_Z) \to H^0(Y, S^k\Omega^1_Y \otimes L \otimes M_Z). \tag{6}$$

Sequence (2), tensored with $L \boxtimes M_Z$ remains exact[1]. Hence, as above, the domain of these k-th gaussian maps is the kernel of the previous ones:

$$\gamma^k_{L,M_Z} : ker\gamma^{k-1}_{L,M_Z} \to H^0(S^k\Omega^1_Y \otimes L \otimes M_Z). \tag{7}$$

In this note we will deal with the following setup: X an abelian surface, $C \subset X$ a smooth and irreducible curve of genus $g \geq 2$. The line bundles on X and C will be respectively $L = \mathcal{O}_X(C)$ (necessarily ample) and K_C. The curve-surface gaussian maps of order ≤ 2 associated to these data are the following. The curve-surface multiplication map

$$\gamma^0_{X,C} : H^0(X, \mathcal{O}_X(C)) \otimes H^0(C, K_C) \to H^0(K^2_C) \tag{8}$$

and the first and second curve-surface gaussian maps

$$\gamma^1_{X,C} : H^0(X \times X, I_{\Delta_X} \otimes \mathcal{O}_X(C) \boxtimes K_C) \to H^0(\Omega^1_X \otimes K^2_C) \tag{9}$$

$$\gamma^2_{X,C} : H^0(X \times X, I^2_{\Delta_X} \otimes \mathcal{O}_X(C) \boxtimes K_C) \to H^0(S^2\Omega^1_X \otimes K^2_C) \tag{10}$$

[1]To see this, since q^*M_Z is locally free on $Y \times Z$, and $I^k_{\Delta_Y}/I^{k+1}_{\Delta_Y}$ is locally free on Δ_Y, it suffices to show that $tor_1^{\mathcal{O}_{Y\times Y}}(\mathcal{O}_{\Delta_Y}, \mathcal{O}_{Y\times Z}) = 0$. One calculates such tor tensoring by $\mathcal{O}_{\Delta_Y}$ the exact sequence $0 \to I_{Y\times Z} \to \mathcal{O}_{Y\times Y} \to \mathcal{O}_{Y\times Z} \to 0$, obtaining

$$0 \to tor_1^{\mathcal{O}_{Y\times Y}}(\mathcal{O}_{\Delta_Y}, \mathcal{O}_{Y\times Z}) \to I_{Y\times Z} \otimes \mathcal{O}_{\Delta_Y} \to \mathcal{O}_{\Delta_Y} \to \mathcal{O}_{\Delta_Z} \to 0,$$

where Δ_Z is the diagonal of Z in $Y \times Z$. A calculation in local coordinates shows that $I_{Y\times Z} \otimes \mathcal{O}_{\Delta_Y}$ is the ideal sheaf of Δ_Z in Δ_Y. Therefore $tor_1^{\mathcal{O}_{Y\times Y}}(\mathcal{O}_{\Delta_Y}, \mathcal{O}_{Y\times Z}) = 0$.

2.3. **First gaussian maps and vector bundles.** It is technically useful to see the gaussian maps (1) and (6) defined as the H^0 of maps of coherent sheaves on the variety Y, rather than on the cartesian product. This is achieved as follows: let p and q the two projections of $Y \times Y$. Applying p_* to the exact sequences (2) tensored by M one gets the exact sequences

$$0 \to p_*(I_{\Delta_Y}^{k+1} \otimes q^*M) \to p_*(I_{\Delta_Y}^{k} \otimes q^*M) \xrightarrow{\varphi^k} S^k\Omega^1_Y \otimes M \tag{11}$$

The gaussian maps $\gamma^k_{L,M}$ of (1) are obtained by tensoring with L and taking $H^0(L \otimes \varphi^k)$. The same with the gaussian map γ^k_{L,M_Z}.

Let us spell out how the gaussian maps γ^1_C, $\gamma^1_{X,C}$, look like in this setting. Let R_C be the the kernel of the evaluation map of K_C:

$$0 \to R_C \xrightarrow{f} H^0(K_C) \otimes \mathcal{O}_C \to K_C \to 0 \tag{12}$$

(i.e. sequence (11) for $Y = C$, $M = K_C$, $k = 0$). By (11) (same setting) for $k = 1$ we have the natural map

$$R_C \xrightarrow{g} K_C^2. \tag{13}$$

Tensoring with K_C and taking H^0 one obtains the Wahl map

$$\gamma^1_C : H^0(R_C \otimes K_C) \to H^0(K_C^3). \tag{14}$$

Next, let G be the kernel of the evaluation map of K_C, *seen as a sheaf on* X,

$$0 \to G \to H^0(K_C) \otimes \mathcal{O}_X \to K_C \to 0 \tag{15}$$

(i.e. sequence (11) for $k = 0$, $Y = X$, $M = K_C$). From (11) for $k = 1$ (same setting) one has the map

$$G \xrightarrow{h} \Omega^1_X \otimes K_C. \tag{16}$$

It is easily seen that G is a locally free sheaf on X, which is sometimes called a Lazarsfeld's bundle, since this type of construction was systematically used in [10]. Tensoring with K_C and taking H^0 one obtains the first curve-surface gaussian map

$$\gamma^1_{X,C} : H^0(G \otimes K_C) \to H^0(\Omega^1_X \otimes K_C^2). \tag{17}$$

Lemma 2.1. *Restricting G to C, one obtains the exact commutative diagram*

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & R_C^2 \arrow[r, "="] \arrow[d, "\alpha"] & R_C^2 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_C \arrow[r] \arrow[d, "="] & G_{|C} \arrow[r] \arrow[d, "h"] & R_C \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & \mathcal{O}_C \arrow[r] & K_C \otimes \Omega_X^1 \arrow[r] & K_C^2 \arrow[r] & 0
\end{tikzcd}$$

where R_C^2 is the kernel of g (see also (20) below).

Proof. Restricting sequence (15) to C one has

$$0 \to tor_1^{\mathcal{O}_X}(K_C, \mathcal{O}_C) \to G_{|C} \to H^0(K_C) \otimes \mathcal{O}_C \to K_C \to 0,$$

where $tor_1^{\mathcal{O}_X}(K_C, \mathcal{O}_C) \cong K_C \otimes tor_1^{\mathcal{O}_X}(\mathcal{O}_C, \mathcal{O}_C) \cong \mathcal{O}_C$. $\square$

2.4. **Second gaussian maps and vector bundles.** Next, the second gaussian maps γ_C^2 and $\gamma_{X,C}^2$. One has the exact sequence

$$0 \to I_{\Delta_Y}^2 \to \mathcal{O}_{Y \times Y} \to \mathcal{O}_{\Delta_Y^2} \to 0 \tag{18}$$

where Y is either the surface X or the curve C and Δ_Y^2 denotes the first infinitesimal neighborhood.

If $Y = C$, tensoring sequence (18) with q^*K_C and applying p_* one gets the exact sequence

$$0 \to R_C^2 \xrightarrow{f'} H^0(K_C) \otimes \mathcal{O}_C \xrightarrow{ev} P_C(K_C) \tag{19}$$

where

$$R_C^2 = p_*(q^*(K_C) \otimes I_{\Delta_C}^2), \tag{20}$$

and

$$P_C(K_C) = p_*(q^*(K_C) \otimes \mathcal{O}_{\Delta_C^2})$$

is the bundle of principal parts of K_C.

Remark 2.2. The commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & R^2_C & \xrightarrow{=} & R^2_C & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & R_C & \to & H^0(K_C)\otimes\mathcal{O}_C & \to & K_C & \to & 0 \\ & & \downarrow g & & \downarrow ev & & \downarrow = & & \\ 0 & \to & K^2_C & \to & P_C(K_C) & \to & K_C & \to & 0 \end{array}$$

shows that the map g of (13) is surjective if and only if the evaluation map ev is surjective. This is in turn equivalent to the immersivity of the canonical map, which holds if and only if C is non-hyperelliptic.

Sequence (11) for $k = 2$, $Y = C$ and $M = K_C$ provides the natural map

$$R^2_C \xrightarrow{g'} K^3_C. \tag{21}$$

Tensoring with K_C and taking H^0 one obtains the second Wahl map

$$\gamma^2_C : H^0(R^2_C \otimes K_C) \to H^0(K^4_C), \tag{22}$$

Finally, let us work out the second curve-surface gaussian map. Let us consider the sequence (18) with Y equal to the surface X. Tensoring sequence (18) with $q^*(K_C)$ and applying p_* one gets

$$0 \to G^2 \to H^0(K_C) \otimes \mathcal{O}_X \xrightarrow{ev} P_X(K_C) \tag{23}$$

where $G^2 = p_*(q^*(K_C) \otimes I^2_{\Delta_X})$. The sheaf $P_X(K_C) = p_*(q^*(K_C) \otimes \mathcal{O}_{\Delta^2_X})$ could be referred to as the "the sheaf of principal parts of K_C on X". Applying p_* to the exact sequence

$$0 \to I_{\Delta_X}/I^2_{\Delta_X} \otimes q^*(K_C) \to \mathcal{O}_{\Delta^2_X} \otimes q^*(K_C) \to \mathcal{O}_{\Delta_X} \otimes q^*(K_C) \to 0$$

one sees that $P_X(K_C)$ sits into the exact sequence (of $\mathcal{O}_X$-modules)

$$0 \to \Omega^1_X \otimes K_C \to P_X(K_C) \to K_C \to 0.$$

Sequence (11) with $k = 2$, $Y = X$ and $M = K_C$ provides the map

$$G^2 \xrightarrow{h'} S^2\Omega^1_X \otimes K_C \tag{24}$$

Tensoring with K_C and taking H^0 one obtains the second curve-surface gaussian map

$$\gamma^2_{X,C} : H^0(G^2 \otimes K_C) \to H^0(S^2\Omega^1_X \otimes K^2_C). \tag{25}$$

Let us consider the exact sequence

$$0 \to \Omega^1_X \otimes K^{-1}_C \to S^2\Omega^1_{X|C} \to K^2_C \to 0 \tag{26}$$

obtained taking symmetric products in the cotangent sequence. The following Lemma will be useful in the sequel

Lemma 2.3. *Assume that C is non-hyperelliptic. Restricting G^2 to C one gets the commutative exact diagram*

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_{X|C} \arrow[r] \arrow[d, "="] & G^2_{|C} \arrow[r] \arrow[d, "h'"] & R^2_C \arrow[r] \arrow[d, "g'"] & 0 \\
0 \arrow[r] & \Omega^1_{X|C} \arrow[r] & K_C \otimes S^2\Omega^1_X \arrow[r] & K^3_C \arrow[r] & 0
\end{tikzcd}$$

Proof. We have the commutative exact diagram
(27)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & G(-C) \arrow[r] \arrow[d] & G^2 \arrow[r] \arrow[d] & R^2_C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(K_C) \otimes \mathcal{O}_X(-C) \arrow[r] \arrow[d, "ev(-C)"] & H^0(K_C) \otimes \mathcal{O}_X \arrow[r] \arrow[d, "ev"] & H^0(K_C) \otimes \mathcal{O}_C \arrow[r] \arrow[d, "ev"] & 0 \\
0 \arrow[r] & \mathcal{O}_C \arrow[r] \arrow[d] & P_X(K_C) \arrow[r] & P_C(K_C) \arrow[r] & 0 \\
 & 0 & & &
\end{tikzcd}$$

Restricting the top row to C one obtains

$$0 \to tor_1^{\mathcal{O}_S}(R^2_C, \mathcal{O}_C) \xrightarrow{a} G_{|C} \otimes K_C^{-1} \xrightarrow{b} G^2_{|C} \to R^2_C \to 0,$$

where $tor_1^{\mathcal{O}_S}(R^2_C, \mathcal{O}_C) \cong R^2_C \otimes tor_1^{\mathcal{O}_S}(\mathcal{O}_C, \mathcal{O}_C) \cong R^2_C \otimes K_C^{-1}$.

We claim that the map $a : R^2_C \otimes K_C^{-1} \to G_{|C} \otimes K_C^{-1}$ is the map α of the diagram of Lemma 2.1 tensored with K_C^{-1}. Granting this for the moment, let us conclude the proof. Again in the diagram of Lemma 2.1 we have that, since C is assumed to be non-hyperelliptic, the map g is surjective (Remark 2.2). Therefore also the map h is surjective. By the Claim, this means that $Im(b) = \Omega^1_{X|C}$. This proves the Lemma. To prove the Claim, we consider

the commutative diagram
(28)

$$\begin{tikzcd}
0 \arrow[r] & R_C^2 \otimes K_C^{-1} \arrow[r, "a"] & G_{|C} \otimes K_C^{-1} \arrow[r, "b"] \arrow[d] & G^2_{|C} \arrow[r] \arrow[d] & R_C^2 \arrow[r] & 0 , \\
 & & G_{|C} \arrow[r, "="] \arrow[d, "="] & G_{|C} \arrow[d] & & \\
0 \arrow[r] & R_C^2 \arrow[r, "\alpha"] & G_{|C} \arrow[r, "h"] \arrow[d] & \Omega^1_{X|C} \otimes K_C \arrow[r] \arrow[d] & 0 & \\
 & & 0 & 0 & &
\end{tikzcd}$$

It follows that $a = \alpha \otimes K_C^{-1}$. $\square$

Remark 2.4. In the same way one proves that, if C is hyperelliptic, restricting G^2 to C one obtains the following commutative exact diagram (notation as in the above proof)

$$\begin{tikzcd}
0 \arrow[r] & Im(h) \otimes K_C^{-1} \arrow[r] \arrow[d] & G^2_{|C} \arrow[r] \arrow[d, "h'"] & R_C^2 \arrow[r] \arrow[d, "g'"] & 0 \\
0 \arrow[r] & \Omega^1_{X|C} \arrow[r] & K_C \otimes S^2\Omega^1_X \arrow[r] & K_C^3 \arrow[r] & 0
\end{tikzcd} \tag{29}$$

3. The first Wahl map of curves on abelian surfaces

According to Beauville and Mérindol ([2]), we focus on the extension class

$$e \in \mathrm{Ext}^1(K_C, K_C^{-1})$$

of the cotangent sequence

$$0 \to K_C^{-1} \to \Omega^1_{X|C} \to K_C \to 0. \tag{30}$$

Since the surface X is abelian, the cotangent bundle is trivial. This implies that $e \neq 0$. Via Serre duality, $\mathrm{Ext}^1(K_C, K_C^{-1})$ is identified to $H^0(K_C^3)^\vee$. Under this identification we have

Lemma 3.1. *Let X be an abelian surface and let $C \subset X$ be a smooth and irreducible curve. Assume that the curve-surface multiplication map*

$$\gamma^0_{X,C} : H^0(X, \mathcal{O}_X(C)) \otimes H^0(C, K_C) \to H^0(K_C^2)$$

is surjective. Then

$$e \notin \mathrm{Ann}(Im(\gamma^1_C)).$$

Proof. By functoriality of Serre duality, the Serre-dual gaussian map

$$\gamma_C^{1\vee} : \mathrm{Ext}^1(K_C^2, \mathcal{O}_C) \to \mathrm{Ext}^1(R_C, \mathcal{O}_C)$$

is obtained applying $\mathrm{Ext}^1(\cdot, \mathcal{O}_C)$ to the map g of (13). We will prove that $\gamma_C^{1\vee}(e)$ is non-zero. To this purpose, applying $\mathrm{Ext}^1(\cdot, \mathcal{O}_C)$ to the map f of sequence (12), one gets the map

$$\psi : \mathrm{Hom}(H^0(K_C), H^1(\mathcal{O}_C)) \cong \mathrm{Ext}^1(H^0(K_C) \otimes \mathcal{O}_C, \mathcal{O}_C) \to \mathrm{Ext}^1(R_C, \mathcal{O}_C).$$

Now let

$$\delta : H^0(K_C) \to H^1(\mathcal{O}_C)$$

be the composition of the coboundary map of the standard exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to K_C \to 0 \tag{31}$$

with the natural injection $i : H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_C)$. It follows form its definition that, since the surface X is abelian, the map δ is non-zero (its image is $i(H^1(\mathcal{O}_X))$. The key point is the following

Claim 3.2. $\gamma_C^{1\vee}(e) = \psi(\delta)$.

Admitting the Claim for the moment, let us finish the proof.
Applying $\mathrm{Ext}^1(\cdot, \mathcal{O}_C)$ to sequence (12), one sees that the kernel of ψ is the image of the map

$$\gamma_C^{0\vee} : \mathrm{Ext}^1(K_C, \mathcal{O}_C) \to \mathrm{Hom}(H^0(K_C), H^1(\mathcal{O}_C)), \tag{32}$$

which is the dual of the multiplication map (3) for $X = C$ and $L = K_C$. Therefore, $\gamma_C^{1\vee}(e) = 0$ means, by the Claim, that $\delta \in \ker(\psi) = Im(\gamma_C^{0\vee})$. The restriction of the map δ to the linear series

$$V = Im(H^0(X, \mathcal{O}_X(C)) \to H^0(K_C))$$

is zero, by definition. Hence, if $\gamma_C^{1\vee}(e) = 0$, then there exists an element $\eta \in \mathrm{Ext}^1(K_C, \mathcal{O}_C)$ such that $\gamma^{0\vee}_C(\eta) = \delta$, and η belongs to the kernel of the map

$$\mathrm{Ext}^1(K_C, \mathcal{O}_C) \to \mathrm{Hom}(H^0(\mathcal{O}_X(C)), H^1(\mathcal{O}_C)).$$

Now this last map is the dual of the curve-surface multiplication map $\gamma^0_{X,C}$: $H^0(\mathcal{O}_X(C)) \otimes H^0(K_C) \to H^0(K_C^2)$. Since δ is non-zero, η is non-zero. But this in contrast with the assumption that the curve-surface multiplication map is surjective.

Proof of the Claim. Lemma 2.1 shows that $\gamma_C^{1\vee}(e)$ is the extension class of the sequence

$$0 \to \mathcal{O}_C \to G_{|C} \to R_C \to 0.$$

Next, we compute $\psi(\delta)$. The exact sequence (31) yields naturally the extension

$$0 \to \mathcal{O}_X \to E \to H^0(K_C) \otimes \mathcal{O}_X \to 0 \tag{33}$$

sitting in the following commutative and exact diagram

(34)
$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & G \arrow[r, "="] \arrow[d] & G \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d, "="] & E \arrow[r] \arrow[d] & H^0(K_C)\otimes\mathcal{O}_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{O}_X(C) \arrow[r] \arrow[d] & K_C \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

This shows that δ is the extension class of the exact sequence

$$0 \to \mathcal{O}_C \to E_{|C} \to H^0(K_C)\otimes\mathcal{O}_C \to 0 \tag{35}$$

obtained by restriction of (33) to C. The commutative diagram with exact rows

(36)
$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_C \arrow[r] \arrow[d, "="] & G_{|C} \arrow[r] \arrow[d] & R_C \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & \mathcal{O}_C \arrow[r] & E_{|C} \arrow[r] & H^0(K_C)\otimes\mathcal{O}_C \arrow[r] & 0
\end{tikzcd}$$

proves that $\psi(\delta) = (\gamma^1_C)^\vee(e)$, i.e. the Claim. $\square$

As a consequence we get the main result of this section (see Theorem B of the Introduction)

Theorem 3.3. *Let X be an abelian surface and let $C \subset X$ be a smooth and irreducible curve. Assume that the curve-surface multiplication map*

$$\gamma^0_{X,C} : H^0(X, \mathcal{O}_X(C)) \otimes H^0(C, K_C) \to H^0(K_C^2)$$

is surjective. Then the natural map

$$\operatorname{coker} \gamma^1_X \to \operatorname{coker} \gamma^1_C$$

is surjective. In particular if, in addition, the Gaussian map γ^1_X on the surface X is surjective then the Gaussian map γ^1_C on the curve C is surjective.

Proof. We have the natural commutative diagram

$$\begin{tikzcd}
\wedge^2 H^0(X,\mathcal{O}_X(C)) \arrow[r, "\gamma^1_X"] \arrow[dd] & H^0(\Omega^1_X\otimes\mathcal{O}_X(2C)) \arrow[dr, "q_1"] & \\
 & & H^0(\Omega^1_{X|C}\otimes K_C^2) \arrow[dl, "q_2"] \\
\wedge^2 H^0(K_C) \arrow[r, "\gamma^1_C"'] & H^0(K_C^3) &
\end{tikzcd}$$

By Serre duality the cotangent extension class e is identified to the linear functional $f_e : H^0(K_C^3) \to H^1(K_C) \cong \mathbb{C}$ defined by the coboundary map of contangent sequence tensored with K_C^2:

$$0 \to K_C \to \Omega^1_{X|C} \otimes K_C^2 \to K_C^3 \to 0.$$

The map q_1 is surjective. The theorem follows since, by Lemma 3.1 the one-codimensional subspace $Im\,(q_2\circ q_1) = Im\,(q_2)$ does not contain the image of the the Gaussian map γ^1_C. □

Remark 3.4. (a) If X is a K3 surface, instead of abelian, Claim 3.2 recovers Beauville-Mérindol's theorem, asserting the opposite happens, namely that $e \in Ann({\gamma^1_C}^\vee)$, and therefore, (at least if $e \neq 0$) γ^1_C is not surjective ([2]).
(b) Let V be the image of the natural map $H^0(\mathcal{O}_X(C)) \to H^0(K_C)$, and let

$$\gamma^1_{V,C} : \wedge^2 V \to H^0(K_C^3)$$

be the restricted Wahl map. By a similar argument, Claim 3.2 proves that $e \in Ann(Im\,\gamma^1_{V,C})$. Hence $\gamma^1_{V,C}$ is not surjective.

Note that – by an immediate computation – the assumption of Theorem 3.3 holds as soon as one as the multiplication map on X:

$$\gamma^0_X : S^2H^0(X,\mathcal{O}_X(C)) \to H^0(X,\mathcal{O}_X(2C))$$

is surjective. This is known to hold when:

- $\mathcal{O}_X(C)$ is a power of order at least 3 of a (necessarily ample) line bundle (Koizumi, [9] Th. 7.3.1);

- $\mathcal{O}_X(C)$ is a second power, and no point of the finite group $K(\mathcal{O}_X(C))$ is a base point of a symmetric line bundle algebraically equivalent to $\mathcal{O}_C(C)$ (Ohbuchi, [9] Prop. 7.2.3). This result and the previous one hold for abelian varieties of any dimension.

- $\mathcal{O}_X(C)$ is not a power (i.e. $\mathcal{O}_X(C)$ is of type $(1,d)$), it is birational, and $d = g(C) - 1 \geq 7$, if d is odd, $d \geq 14$, if d is even, (Lazarsfeld, [11]).

- $\mathcal{O}_X(C)$ of type $(1,d)$, the Néron-Severi group $NS(X)$ is generated by $c_1(\mathcal{O}_X(C))$ and $d \geq 7$ (Iyer, [8]).

Finally, let us focus on surjectivity of the first gaussian map on X:

$$\gamma_X^1 : \wedge^2 H^0(X, \mathcal{O}_X(C)) \to H^0(\Omega_X^1(2C)). \tag{37}$$

Unlike for multiplication maps, effective surjectivity criteria for gaussian maps of polarizations of type $(1,d)$ on abelian surfaces are not available at present. The only result we are aware of is the analogue of Koizumi's theorem, asserting that, if $\mathcal{O}_X(C)$ is at least a 5-th power, then the gaussian map (37) is surjective ([14] Th.2.1). Therefore, as a consequence of Theorem 3.3, we have

Corollary 3.5. *Let X be an abelian surface and let $\mathcal{L}$ be an ample line bundle on X, and let $k \geq 5$. For all smooth and irreducible curves $C \in |\mathcal{L}^k|$ the Wahl map*

$$\gamma_C^1 : \wedge^2 H^0(K_C) \to H^0(K_C^3)$$

is surjective.

It is worth to note that suitable surjectivity criteria for gaussian maps on abelian surfaces – analogous to those due to Lazarsfeld and Iyer for multiplication maps – would imply, as in the previous Corollary, the surjectivity of the Wahl map of general curves of any suitably high genus lying on abelian surfaces, thus providing a "without degeneration" proof of the theorem of Ciliberto-Harris-Miranda.

4. The second Wahl map of curves on abelian surfaces

The present Section is entirely devoted to the proof of Theorem A.

Tensoring the symmetric square of the cotangent sequence (26) with K_C^2 one gets

$$0 \to \Omega_X^1 \otimes K_C \cong K_C^{\oplus 2} \to S^2\Omega^1_{X|C} \otimes K_C^2 \to K_C^4 \to 0 \tag{38}$$

whose coboundary map

$$f_{e'} : H^0(K_C^4) \to H^1(\Omega_X^1 \otimes K_C) \cong H^1(K_C)^{\oplus 2} \cong \mathbb{C}^{\oplus 2}$$

is identified, by Serre duality, to the extension class $e' \in \operatorname{Ext}^1(K_C^3, \Omega^1_{X|C})$ of sequence (26). The first part of the statement of Theorem A is equivalent to:

$$f_{e'} \circ \gamma_C^2 = 0. \tag{39}$$

As for the case of first gaussian maps, it is easier to work in the dual setting. Applying $\operatorname{Ext}^1(\cdot, \Omega^1_{X|C})$ to the map g' of (21) one gets the map

$$\phi : \operatorname{Ext}^1(K_C^3, \Omega^1_{X|C}) \to \operatorname{Ext}^1(R_C^2, \Omega^1_{X|C})$$

(which is identified to two copies of the dual map of the second Wahl map) and it is easily seen that (39) is equivalent to the fact that

$$\phi(e') = 0. \tag{40}$$

Applying $\mathrm{Ext}^1(\cdot, \Omega^1_{X|C})$ to the map f' of (19) we get the map

$$\psi : \mathrm{Hom}(H^0(K_C), H^1(\Omega^1_{X|C})) = \mathrm{Ext}^1(H^0(K_C)\otimes\mathcal{O}_C, \Omega^1_{X|C}) \to \mathrm{Ext}^1(R^2_C, \Omega^1_{X|C}).$$

Now let us denote by $\tilde{\delta}$ the composition of the coboundary map $H^0(K_C) \to H^1(\mathcal{O}_X)$ of the standard exact sequence (31), and the map $H^1(d) : H^1(\mathcal{O}_X) \to H^1(\Omega^1_{X|C})$, induced by the derivation $d : \mathcal{O}_X \to \Omega^1_X$. Note that $H^1(d)$ is the zero map since the Hodge-Frölicher spectral sequence degenerates at level of E_1.

Claim 4.1. $\phi(e') = \psi(\tilde{\delta}) = 0$

The first part of Theorem A, i.e. (39), follows immediately since $\tilde{\delta} = 0$.

Proof of the Claim. Assume that C is non-hyperelliptic. Lemma 2.3 shows that $\phi(e')$ is the class of the sequence

$$0 \to \Omega^1_{X|C} \to G^2_{|C} \to R^2_C \to 0$$

Next, we compute $\psi(\tilde{\delta})$. The zero map $H^1(d) \in \mathrm{Hom}(H^1(\mathcal{O}_X), H^1(\Omega^1_X))$ can be seen as the class of the extension

$$0 \to \Omega^1_X \to H := R^1q_*(I^2_{\Delta_X}) \to H^1(\mathcal{O}_X) \otimes \mathcal{O}_X \to 0 \tag{41}$$

obtained applying q_* to the exact sequence (2) with $k = 1$. In fact, by Leray spectral sequence the coboundary map of (41) can be identified with the map in cohomology:

$$H^1(X, \mathcal{O}_X) \overset{F}{\tilde{\to}} H^1(X \times X, I_{\Delta_X}) \xrightarrow{G} H^1(X \times X, I_{\Delta_X}/I^2_{\Delta_X}) \cong H^1(\Omega^1_X)$$

induced by the exact sequence (2) with $k = 1$. Künneth formula gives the isomorphism $H^1(X, \mathcal{O}_X) \oplus H^1(X, \mathcal{O}_X) \tilde{\to} H^1(X \times X, \mathcal{O}_{X\times X})$, $(\alpha, \beta) \mapsto p^*(\alpha) + q^*(\beta)$. Therefore the map $F : H^1(X, \mathcal{O}_X) \to H^1(X \times X, I_{\Delta_X})$ is given by $\alpha \mapsto p^*(\alpha) - q^*(\alpha)$, hence $G \circ F = H^1(d)$, since the map $d : \mathcal{O}_X \to I_{\Delta_X}/I^2_{\Delta_X}$ is exactly the map $\alpha \mapsto p^*(\alpha) - q^*(\alpha)$.

We have the natural exact diagram

(42)
$$\begin{tikzcd}
 & 0 \arrow[r] & R^2_X \arrow[r] \arrow[d] & H^0(\mathcal{O}_X(C)) \otimes \mathcal{O}_X \arrow[r] \arrow[d] & P_X(\mathcal{O}_X(C)) \arrow[d] \\
 & 0 \arrow[r] & G^2 \arrow[r] \arrow[d] & H^0(K_C) \otimes \mathcal{O}_X \arrow[r] \arrow[d] & P_X(K_C) \arrow[d] \\
0 \arrow[r] & \Omega^1_X \arrow[r] & H \arrow[r] & H^1(\mathcal{O}_X) \otimes \mathcal{O}_X \arrow[r] & 0
\end{tikzcd}$$

where $R^2_X = q_*(I^2_{\Delta_X} \otimes p^*(\mathcal{O}_X(C)))$. The first column of the diagram is obtained from the exact sequence

$$0 \to p^*(\mathcal{O}_X) \to p^*(\mathcal{O}_X(C)) \to p^*(K_C) \to 0$$

tensoring with $I^2_{\Delta_X}$ and applying q_*. Notice that tensoring with $I^2_{\Delta_X}$, the above exact sequence remains exact, since

$$tor_1^{\mathcal{O}_{X\times X}}(p^*(K_C), I^2_{\Delta_X}) = tor_2^{\mathcal{O}_{X\times X}}(p^*(K_C), I_{\Delta_X}/I^2_{\Delta_X}) = 0.$$

Restricting the two bottom rows to C one gets

(43)
$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_{X|C} \arrow[r] \arrow[d, "="] & G^2_{|C} \arrow[r] \arrow[d] & R^2_C \arrow[r] \arrow[d, "\lambda"] & 0 \\
0 \arrow[r] & \Omega^1_{X|C} \arrow[r] & H_{|C} \arrow[r] & H^1(\mathcal{O}_X) \otimes \mathcal{O}_C \arrow[r] & 0
\end{tikzcd}$$

where λ is the composition

$$R^2_C \stackrel{g}{\to} H^0(K_C) \otimes \mathcal{O}_C \stackrel{\delta\otimes\mathcal{O}_C}{\to} H^1(\mathcal{O}_X) \otimes \mathcal{O}_C.$$

Hence $\phi(e') = \psi(\tilde{\delta})$. This proves Claim 4.1 if C is non-hyperelliptic. If C is hyperelliptic the same argument applies, using the diagram of Remark 2.4 instead of the one of Lemma 2.3. This concludes the proof of Claim 4.1, and hence of the first part of the statement of Theorem A.

The last part of the statement follows as in Theorem 3.3 from the commutative diagram

(44)
$$\begin{tikzcd}
I_2(\mathcal{O}_X(C)) \arrow[r, "\gamma^2_X"] \arrow[dd] & H^0(S^2\Omega^1_X \otimes \mathcal{O}_X(2C)) \arrow[dr] & \\
& & H^0(S^2\Omega^1_{X|C} \otimes K^2_C) \arrow[dl] \\
I_2(K_C) \arrow[r, "\gamma^2_C"'] & H^0(K^4_C) &
\end{tikzcd}$$

□

Finally, it is known that, if $\mathcal{O}_X(C)$ is at least a 7-power of a (necessarily ample) line bundle on X, then the second gaussian map γ^2_X is surjective (this is part of a general result on surjectivity of higher gaussian maps of any order for powers of ample line bundles on abelian varieties, [14] Th. 2.2). Hence we have the following

Corollary 4.2. *Let X be an abelian surface, let $\mathcal{L}$ be an ample line bundle on X and let $k \geq 7$. Then, for every smooth and irreducible curve $C \in |\mathcal{L}^k|$,*

the image of second Wahl map

$$\gamma_C^2 : I_2(K_C) \to H^0(K_C^4)$$

is the 2-codimensional subspace $S^2H^0(\Omega^1_X) \cdot H^0(K_C^2)$.

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